Thermal buckling of functionally graded plates using a n-order four variable refined theory

Z. Abdelhak*1, L. Hadji1,2, T.H. Daouadj1,2 and E.A Bedia2

1Université Ibn Khaldoun, BP 78 Zaaroura, 14000 Tiaret, Algérie.
2Laboratoire des Matériaux & Hydrologie, Université de Sidi Bel Abbes, 22000 Sidi Bel Abbes, Algérie.

(Received December 2, 2014, Revised January 28, 2015, Accepted February 25, 2015)

Abstract. This paper presents a simple n-order four variable refined theory for buckling analysis of functionally graded plates. By dividing the transverse displacement into bending and shear parts, the number of unknowns and governing equations of the present theory is reduced, and hence, makes it simple to use. The present theory is variationally consistent, uses the n-order polynomial term to represent the displacement field, does not require shear correction factor, and eliminates the shear stresses at the top and bottom surfaces. A power law distribution is used to describe the variation of volume fraction of material compositions. Equilibrium and stability equations are derived based on the present n-order refined theory. The non-linear governing equations are solved for plates subjected to simply supported boundary conditions. The thermal loads are assumed to be uniform, linear and non-linear distribution through-the-thickness. The effects of aspect and thickness ratios, gradient index, on the critical buckling are all discussed.

Keywords: nth-order four variable refined theory; functionally graded plates; thermal buckling

1. Introduction

Functionally graded materials (FGMs) are new inhomogeneous materials which have widely used in many engineering applicants such as nuclear reactors and high-speed spacecraft industries (Yamanouchi et al, 1990). The mechanical properties of FGMs vary smoothly and continuously from one surface to the other. Typically these materials are made from a mixture of ceramic and metal or from a combination of different materials. The ceramic constituent of the material provides the high-temperature resistance due to its low thermal conductivity. The ductile metal constituent on the other hand, prevents fracture caused by stresses due to the high temperature gradient in a very short period of time. Furthermore a mixture of ceramic and metal with a continuously varying volume fraction can be easily manufactured (Fukui 1991, Koizumi 1997). With the developments in manufacturing methods (Fukui et al, 1991, Fukui et al, 1997 and El-Hadek et al, 2003) functionally graded materials seem to have great potential in sandwich structures. The analysis of these materials has been considered by many researchers. The functionally graded (FG) plates are commonly used in thermal environments; they can buckle under thermal and mechanical loads. Thus, the buckling analysis of such plates is essential to

*Corresponding Author, Professor, E-mail: had_laz@yahoo.fr

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http://www.techno-press.org/?journal=amr&subpage=7 ISSN: 2234-0912 (Print), 2234-179X (Online)
ensure an efficient and reliable design. Eslami and his co-workers (Javaheri et al. 2002, Samsam et al. 2005) have treated a series of problems relating to the linear buckling of simply supported rectangular FG plates, with and without imperfections, under mechanical and thermal loads. By using an analytical approach, they obtained closed-form expressions for buckling loads. Sohn et al. (2008) dealt with the stabilities of FG panels subjected to combined thermal and aerodynamic loads. The first-order theory was used to simulate supersonic aerodynamic loads acting on the panels. The influence of the material constitution of FG panels on thermal buckling and flutter characteristics was examined. Zenkour et al. (2010a) studied the thermal Buckling Analysis of Ceramic-Metal Functionally Graded Plates. Bouiadjra et al. (2012) developed a four-variable refined plate theory for buckling analysis of functionally graded plates. Song et al. (2013) used a n-order four variable refined theory for bending and free vibration of functionally graded plates. Klouche (2014) studied the bending and free vibration of functionally graded plates by using a n-order four variable refined theory. The n-order four variable refined theory proposed by Song et al. (2013) is based on the assumption that the in-plane and transverse displacements consist of bending and shear components in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments.

The most interesting feature of this theory is uses the n-order polynomial term to represent the displacement field, does not require shear correction factor, and eliminates the shear stresses at the top and bottom surfaces. Although several studies on the buckling of FG plates have been carried out based on variety of plate theories, no studies can be found for the thermal buckling of FG plates based on the refined theory proposed by Song Xiang et al. (2013). Therefore, the aim of this study is to extend the n-order refined theory to the thermal buckling of FG plates. The material properties of FG plate are assumed to vary according to a power law distribution of the volume fraction of the constituents. The n-order four variable refined theory is used to obtain the buckling of the plate under different types of thermal loads. The thermal loads are assumed to be uniform, linear and non-linear distribution through the thickness. Illustrative examples are given so as to demonstrate the efficacies of the theory. The effects of various variables, such as thickness and aspect ratios, gradient index, on the critical buckling are all discussed.

2. Theoretical formulation

2.1 Displacement field and strains

Consider a plate of total thickness \( h \) and composed of functionally graded material through the thickness. It is assumed that the material is isotropic and grading is assumed to be only through the thickness. The \( xy \) plane is taken to be the undeformed mid plane of the plate with the \( z \) axis positive upward from the mid plane. The displacement field of this theory is as follows:

\[
\begin{align*}
    u(x, y, z, t) &= u_0(x, y, t) - z \frac{\partial w_h}{\partial x} - \frac{1}{n} \left( \frac{2}{h} \right)^{n-1} z^n \frac{\partial w_s}{\partial x}, \\
    v(x, y, z, t) &= v_0(x, y, t) - z \frac{\partial w_h}{\partial y} - \frac{1}{n} \left( \frac{2}{h} \right)^{n-1} z^n \frac{\partial w_s}{\partial y}
\end{align*}
\]

\[
n = 3, 5, 7, 9, ...
\]
\( w(x, y, z, t) = w_b(x, y, t) + w_s(x, y, t) \)

where \( u_0 \) and \( v_0 \) are the mid-plane displacements of the plate in the x and y direction, respectively; \( w_b \) and \( w_s \) are the bending and shear components of transverse displacement, respectively.

The non-linear von Karman strain–displacement equations are as follows:

\[
\begin{align*}
\epsilon_x & = \epsilon_{x}^0 + z \epsilon_{x}^{1} + f(z) k_x^1, \\
\epsilon_y & = \epsilon_{y}^0 + z \epsilon_{y}^{1} + f(z) k_y^1, \\
\gamma_{xy} & = \gamma_{xy}^0 + z \gamma_{xy}^{1} + f(z) k_{xy}^1, \\
\gamma_{yz} & = \gamma_{yz}^0 + z \gamma_{yz}^{1} + f(z) k_{yz}^1, \\
\gamma_{xz} & = \gamma_{xz}^0 + z \gamma_{xz}^{1} + f(z) k_{xz}^1, \\
\end{align*}
\]

where

\[
\begin{align*}
\epsilon_{x}^0 & = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_b}{\partial x} + \frac{\partial w_s}{\partial x} \right)^2, \\
\epsilon_{y}^0 & = \frac{\partial v_0}{\partial y} + \frac{1}{2} \left( \frac{\partial w_b}{\partial y} + \frac{\partial w_s}{\partial y} \right)^2, \\
k_x^1 & = \frac{\partial^2 w_b}{\partial x^2}, \\
k_y^1 & = \frac{\partial^2 w_s}{\partial y^2}, \\
k_{xy}^1 & = -2 \frac{\partial^2 w_b}{\partial x \partial y}, \\
k_{yz}^1 & = -2 \frac{\partial^2 w_s}{\partial y \partial z}, \\
k_{xz}^1 & = -2 \frac{\partial^2 w_s}{\partial x \partial z}, \\
\end{align*}
\]

\[
f(z) = \frac{z^n}{n} \left( \frac{2}{h} \right)^{n-1}; \text{ and } g(z) = 1 - \frac{df(z)}{dz} = 1 - \left( \frac{z^n}{z} \right) \left( \frac{2}{h} \right)^{n-1}
\]

2.2 Constitutive relations

Consider a functionally graded plate, which is made from a mixture of ceramics and metals. The plate is subjected to a thermal load \( T(x, y, z) \). It is assumed that the composition properties of FGM vary through the thickness of the plate.

The variation of material properties can be expressed as:

\[
P(z) = P_t + (P_b - P_t)V_t
\]

where \( P \) denotes a generic material property like modulus and \( P_t \) and \( P_b \) denote the corresponding properties of the top and bottom faces of the plate, respectively. Also \( V_t \) in Eq. (4) denotes the volume fraction of the top face constituent and follows a simple power-law as:
\[ V_t = \left( \frac{z}{h} + \frac{1}{2} \right)^k \]  

(5)

where \( k \) \((0 \leq k \leq \infty)\) is a parameter that dictates the material variation profile through the thickness. Here we assume that moduli \( E, G \) and the coefficient of thermal expansion \( \alpha \) vary according to Eq. (4) and the Poisson’s ratio \( \nu \) is assumed to be a constant.

The linear constitutive relations are:

\[
\begin{bmatrix}
\sigma_x

\sigma_y

\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0

Q_{12} & Q_{22} & 0

0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x - \alpha T

\varepsilon_y - \alpha T

\gamma_{xy}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\tau_{yz}

\tau_{zx}
\end{bmatrix} =
\begin{bmatrix}
Q_{44} & 0

0 & Q_{55}
\end{bmatrix}
\begin{bmatrix}
\gamma_{yz}

\gamma_{zx}
\end{bmatrix}
\]

(6)

where \((\sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \tau_{zx})\) and \((\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})\) are the stress and strain components, respectively. Using the material properties defined in Eq. (4), stiffness coefficients, \( Q_{ij} \), can be expressed as

\[
Q_{11} = Q_{22} = \frac{E(z)}{1 - \nu^2}
\]

(7a)

\[
Q_{12} = \frac{\nu E(z)}{1 - \nu^2}
\]

(7b)

\[
Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1 + \nu)}
\]

(7c)

### 2.3 Stability equations

The total potential energy of the FG plate may be written as

\[
U = \frac{1}{2} \int \int \left[ \sigma_x (\varepsilon_x - \alpha T) + \sigma_y (\varepsilon_y - \alpha T) + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx} \right] dx dy dz
\]

(8)

The principle of virtual work for the present problem may be expressed as follows:

\[
\int \int \left[ N_x \delta \sigma_x^0 + N_y \delta \sigma_y^0 + N_{xy} \delta \gamma_{xy}^0 + M_x^b \delta \sigma_x^b + M_y^b \delta \sigma_y^b + M_{xy}^b \delta \gamma_{xy}^b + M_x^s \delta \sigma_x^s + M_y^s \delta \sigma_y^s \right] dx dy dz = 0
\]

(9)

where

\[
\begin{bmatrix}
N_x & N_y & N_{xy}

M_x^b & M_y^b & M_{xy}^b

M_x^s & M_y^s & M_{xy}^s
\end{bmatrix} = \int \left[ \sigma_x, \sigma_y, \tau_{xy} \right] \left[ \begin{array}{c} f(z) \\ f''(z) \end{array} \right] dz
\]

(10a)
\[
\begin{align*}
\left( S^x_{zz}, S^y_{zz} \right) &= \frac{h^2}{2} \int_{-h/2}^{h/2} \left( \tau_{xzz}, \tau_{yzz} \right) k(z) dz. \\
\end{align*}
\]  

Using Eq. (6) in Eq. (10), the stress resultants of the FG plate can be related to the total strains by

\[
\begin{bmatrix}
N \\
M^b \\
M^s
\end{bmatrix} = \begin{bmatrix}
A & B^s & B^b \\
B & D^s & D^b \\
B^s & D^s & H^s
\end{bmatrix} \begin{bmatrix}
\varepsilon \\
\kappa^b \\
\kappa^s
\end{bmatrix} - \begin{bmatrix}
N^T \\
M^{bT} \\
M^{sT}
\end{bmatrix}, \quad S = A^s \gamma
\]  

where

\[
N = \left\{ N_x, N_y, N_{xy} \right\}_t, \quad M^b = \left\{ M^b_x, M^b_y, M^b_{xy} \right\}_t, \quad M^s = \left\{ M^s_x, M^s_y, M^s_{xy} \right\}_t
\]  

\[
N^T = \left\{ N^T_x, N^T_y, 0 \right\}_t, \quad M^{bT} = \left\{ M^{bT}_x, M^{bT}_y, 0 \right\}_t, \quad M^{sT} = \left\{ M^{sT}_x, M^{sT}_y, 0 \right\}_t
\]  

\[
\varepsilon = \left\{ \varepsilon_x, \varepsilon_y, \gamma_{xy} \right\}_t, \quad k^b = \left\{ k^b_x, k^b_y, k^b_{xy} \right\}_t, \quad k^s = \left\{ k^s_x, k^s_y, k^s_{xy} \right\}_t
\]  

\[
A = \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{12} & A_{22} & 0 \\
0 & 0 & A_{66}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_{11} & B_{12} & 0 \\
B_{12} & B_{22} & 0 \\
0 & 0 & B_{66}
\end{bmatrix}, \quad D = \begin{bmatrix}
D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & D_{66}
\end{bmatrix}
\]  

\[
B^s = \begin{bmatrix}
B^s_{11} & B^s_{12} & 0 \\
B^s_{12} & B^s_{22} & 0 \\
0 & 0 & B^s_{66}
\end{bmatrix}, \quad D^s = \begin{bmatrix}
D^s_{11} & D^s_{12} & 0 \\
D^s_{12} & D^s_{22} & 0 \\
0 & 0 & D^s_{66}
\end{bmatrix}, \quad H^s = \begin{bmatrix}
H^s_{11} & H^s_{12} & 0 \\
H^s_{12} & H^s_{22} & 0 \\
0 & 0 & H^s_{66}
\end{bmatrix}
\]  

\[
S = \left\{ S^x_{zz}, S^y_{zz} \right\}_t, \quad \gamma = \left\{ \gamma^x, \gamma^y \right\}_t, \quad A^s = \begin{bmatrix}
A_{14} & 0 \\
0 & A_{55}
\end{bmatrix}
\]  

where \( A_{ij}, \) \( B_{ij}, \) etc., are the plate stiffness, defined by

\[
\begin{align*}
\begin{bmatrix}
A_{11} B_{11} D_{11} B^s_{11} D^s_{11} H^s_{11} \\
A_{12} B_{12} D_{12} B^s_{12} D^s_{12} H^s_{12} \\
A_{66} B_{66} D_{66} B^s_{66} D^s_{66} H^s_{66}
\end{bmatrix} &= \left\{ \begin{bmatrix}
1 \\
1 - v^{(n)}
\end{bmatrix} \right\} dz
\end{align*}
\]
and
\[
\begin{pmatrix}
A_{22}, B_{22}, D_{22}, B'_{22}, D'_{22}, H'_{22}
\end{pmatrix} = \begin{pmatrix}
A_{11}, B_{11}, D_{11}, B'_{11}, D'_{11}, H'_{11}
\end{pmatrix} \tag{13b}
\]
\[
A_{s}^{\varepsilon} = A_{s}^{\varepsilon} = \frac{h^{l/2}}{-h^{l/2}} \frac{E(z)}{2(1-\nu)} \left[ g(z) \right]^{2} dz \tag{13c}
\]

The stress and moment resultants, \(N_{s}^{T} = N_{s}^{T}, M_{s}^{bT} = M_{s}^{bT}\) and \(M_{s}^{ST} = M_{s}^{ST}\) due to thermal loading are defined by
\[
\begin{pmatrix}
N_{s}^{T} \\
M_{s}^{bT} \\
M_{s}^{ST}
\end{pmatrix} = \begin{pmatrix}
\frac{h^{l/2}}{-h^{l/2}} \frac{E(z)}{1-\nu} \left[ f(z) \right]
\end{pmatrix} \tag{14}
\]

The stability equations of the plate may be derived by the adjacent equilibrium criterion. Assume that the equilibrium state of the FG plate under thermal loads is defined in terms of the displacement components \(\left(u_{0}, v_{0}, w_{b}, w_{s}^{l}\right)\). The displacement components of a neighboring stable state differ by \(\left(u_{0}, v_{0}, w_{b}, w_{s}^{l}\right)\) with respect to the equilibrium position. Thus, the total displacements of a neighboring state are
\[
u = u_{0} + u_{b}, \quad v = v_{0} + v_{b}, \quad w_{b} = w_{b}^{0} + w_{b}, \quad w_{s} = w_{s}^{0} + w_{s}^{l}, \tag{15}
\]

where the superscript 1 refers to the state of stability and the superscript 0 refers to the state of equilibrium conditions.

Substituting Eqs. (2) and (15) into Eq. (9) and integrating by parts and then equating the coefficients of \(\partial u_{0}^{l}, \partial v_{0}^{l}, \partial w_{b}^{l}, \partial w_{s}^{l}\), to zero, separately, the governing stability equations are obtained for the shear deformation plate theories as
\[
\begin{align*}
\frac{\partial N_{s}^{l}}{\partial x} + \frac{\partial N_{s}^{l}}{\partial y} &= 0 \\
\frac{\partial N_{s}^{l}}{\partial x} + \frac{\partial N_{s}^{l}}{\partial y} &= 0 \\
\frac{\partial^{2} M_{s}^{bI}}{\partial x^{2}} + 2 \frac{\partial^{2} M_{s}^{bI}}{\partial x \partial y} + \frac{\partial^{2} M_{s}^{bI}}{\partial y^{2}} + \bar{N} &= 0 \\
\frac{\partial^{2} M_{s}^{lI}}{\partial x^{2}} + 2 \frac{\partial^{2} M_{s}^{lI}}{\partial x \partial y} + \frac{\partial^{2} M_{s}^{lI}}{\partial y^{2}} + \frac{\partial S_{s}^{0I}}{\partial x} + \frac{\partial S_{s}^{0I}}{\partial y} + \bar{N} &= 0
\end{align*} \tag{16}
\]

with
\[
\bar{N} = \left[ N_{s}^{0} \frac{\partial^{2}(w_{b}^{0} + w_{s}^{I})}{\partial x^{2}} + N_{s}^{0} \frac{\partial^{2}(w_{b}^{0} + w_{s}^{I})}{\partial y^{2}} + 2 N_{s}^{0} \frac{\partial^{2}(w_{s}^{l} + w_{s}^{l})}{\partial x \partial y} \right] \tag{17}
\]

where the terms \(N_{s}^{0}\) and \(N_{s}^{0}\) are the pre-buckling force resultants obtained as
3. Exact solution for a simply-supported fgm plate

Rectangular plates are generally classified in accordance with the type of support used. We are here concerned with the exact solution of Eqs. (16) for a simply supported FG plate. The following boundary conditions are imposed at the side edges for the present four variable refined plate theory:

\[
v_0^l = w_0^l = w_s^l = \frac{\partial w_s^l}{\partial y} = N_0^l = M_s^{bl} = M_s^{cl} = 0 \quad \text{at} \quad x = 0, a,
\]

\[
u_0^l = w_b^l = w_s^l = \frac{\partial w_s^l}{\partial x} = N_y^l = M_y^{bl} = M_y^{cl} = 0 \quad \text{at} \quad y = 0, b.
\]

The following approximate solution is seen to satisfy both the differential equation and the boundary conditions

\[
\begin{bmatrix}
u_0^l \\ v_0^l \\ w_b^l \\ w_s^l
\end{bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix}
U_{mn}^l \cos(\lambda x) \sin(\mu y) \\
V_{mn}^l \sin(\lambda x) \cos(\mu y) \\
W_{mn}^l \sin(\lambda x) \sin(\mu y) \\
W_{mn}^l \sin(\lambda x) \cos(\mu y)
\end{bmatrix}
\]

where \( U_{mn}^l, V_{mn}^l, W_{mm}^l \) and \( W_{mn}^l \) are arbitrary parameters to be determined, \( \lambda = m \pi / a \) and \( \mu = n \pi / b \) and and \( m \) and \( n \) are mode numbers. Substituting Eq. (20) into Eq. (16), one obtains

\[
[K] \{\Delta\} = \{P\}
\]

where \( \{\Delta\} \) denotes the column

\[
\{\Delta\} = \begin{bmatrix}
U_{mn}^l \\
V_{mn}^l \\
W_{mn}^l \\
W_{mn}^l
\end{bmatrix}
\]

and \([K]\) is the symmetric matrix given by

\[
[K] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{12} & a_{22} & a_{23} & a_{24} \\
a_{13} & a_{23} & a_{33} & a_{34} \\
a_{14} & a_{24} & a_{34} & a_{44}
\end{bmatrix}
\]

For nontrivial solution, the determinant of the coefficient matrix in Eq. (21) must be zero. This gives the thermal buckling load.
3.1 Buckling of FG Plates under Uniform Temperature Rise

The plate initial temperature is assumed to be \( T_i \). The temperature is uniformly raised to a final value \( T_f \) in which the plate buckles. The temperature change is \( \Delta T = T_f - T_i \).

3.2 Buckling of FG Plates Subjected to Graded Temperature Change Across the Thickness

We assume that the temperature of the top surface is \( T_t \) and the temperature varies from \( T_t \) according to the power law variation through-the-thickness, to the bottom surface temperature \( T_b \) in which the plate buckles. In this case, the temperature through-the-thickness is given by

\[
T(z) = \Delta T \left( \frac{z}{h} + \frac{1}{2} \right)^\gamma + T_i
\]

where the buckling temperature difference \( \Delta T = T_f - T_b \) and \( \gamma \) is the temperature exponent \( (0 < \gamma < \infty) \). Note that the value of \( \gamma \) equal to unity represents a linear temperature change across the thickness. While the value of \( \gamma \) excluding unity represents a non-linear temperature change through-the-thickness.

4. Numerical results

In this section, various numerical examples are presented and discussed for verifying the accuracy and efficiency of the present theory in predicting the critical buckling temperature change of simply supported FG plates under uniform, linear and nonlinear thermal loading through the thickness. For the verification purpose, the results obtained by the present four variable refined plate theory are compared with the existing data in the literature.

It is assumed that the functionally graded plate is made of a mixture of aluminum and alumina. The Young modulus, coefficient of thermal expansion and thermal conductivity for aluminum are \( E_m = 70 \text{ GPa} \), \( \alpha_m = 23 \times 10^{-6} \text{/}^\circ \text{C} \), and for alumina are \( E_c = 380 \text{ GPa} \), \( \alpha_c = 7.4 \times 10^{-6} \text{/}^\circ \text{C} \) respectively.

In order to prove the validity of the present formulation, results were obtained for FG plates under uniform, linear and nonlinear thermal loading through the thickness according to all theories.

In Tables 1 and 2 the results of buckling analysis for the plate under uniform temperature rise are presented. These tables show the comparisons of the critical buckling temperature change obtained by the present theory with those given by Javaheri et al. (2002) based on both higher plate theory (HPT) and the classical plate theory (CPT), and Zenkour and Mashat (2010) based on sinusoidal plate theory (SPT). The results of the present theory show very good agreement with HPT and SPT both for thin and thick FG plates.

Table 1 show that the buckling temperature increases by the increase of the aspect ratio \( a/b \) and decreases with increase of the power law index \( (k) \) from 0 to 10. Table 2 shows that the buckling temperature decreases by the increase of the dimension ratio \( a/h \) and the power law
Table 1 Critical buckling temperature of FG plate under uniform temperature rise for different values of power law index $k$ and aspect ratio $a/b$ with $a/b = 100$

<table>
<thead>
<tr>
<th>$k$</th>
<th>Theory</th>
<th>$a/b=1$</th>
<th>$a/b=2$</th>
<th>$a/b=3$</th>
<th>$a/b=4$</th>
<th>$a/b=5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Present (n=3)</td>
<td>17.0894</td>
<td>42.6875</td>
<td>85.2551</td>
<td>144.6490</td>
<td>220.6706</td>
</tr>
<tr>
<td></td>
<td>Present (n=5)</td>
<td>17.0896</td>
<td>42.6888</td>
<td>85.2600</td>
<td>144.6630</td>
<td>220.7033</td>
</tr>
<tr>
<td></td>
<td>Present (n=7)</td>
<td>17.0898</td>
<td>42.6902</td>
<td>85.2658</td>
<td>144.6798</td>
<td>220.7423</td>
</tr>
<tr>
<td></td>
<td>Present (n=9)</td>
<td>17.0900</td>
<td>42.6913</td>
<td>85.2703</td>
<td>144.6927</td>
<td>220.7723</td>
</tr>
<tr>
<td></td>
<td>HPT*</td>
<td>17.08</td>
<td>42.68</td>
<td>85.25</td>
<td>144.64</td>
<td>220.66</td>
</tr>
<tr>
<td></td>
<td>SPT*</td>
<td>17.08</td>
<td>42.68</td>
<td>85.25</td>
<td>144.65</td>
<td>220.28</td>
</tr>
<tr>
<td></td>
<td>CPT*</td>
<td>17.09</td>
<td>42.74</td>
<td>85.49</td>
<td>145.34</td>
<td>222.28</td>
</tr>
<tr>
<td>1</td>
<td>Present (n=3)</td>
<td>7.9400</td>
<td>19.8358</td>
<td>39.6248</td>
<td>67.2506</td>
<td>102.6356</td>
</tr>
<tr>
<td></td>
<td>Present (n=5)</td>
<td>7.9400</td>
<td>19.8363</td>
<td>39.6267</td>
<td>67.2561</td>
<td>102.6484</td>
</tr>
<tr>
<td></td>
<td>Present (n=7)</td>
<td>7.9401</td>
<td>19.8369</td>
<td>39.6289</td>
<td>67.2627</td>
<td>102.6637</td>
</tr>
<tr>
<td></td>
<td>Present (n=9)</td>
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</table>

*Results from Ref (Zenkov et al. 2010)
*Results from Ref (Javaheri et al. 2002b)

Table 2 Critical buckling temperature of square FG plate under uniform temperature rise for different values of power law index $k$ and side-to-thickness ratio $a/h$

<table>
<thead>
<tr>
<th>$k$</th>
<th>Theory</th>
<th>$a/h=10$</th>
<th>$a/h=20$</th>
<th>$a/h=40$</th>
<th>$a/h=60$</th>
<th>$a/h=80$</th>
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<tbody>
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<td>17.09</td>
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</table>
The buckling temperatures for homogeneous plates are considerably higher than those for the FG plates, especially for the comparatively longer and thicker plates. The critical buckling temperature increases by the increase of the dimension ratio $a/h$, decreases by the increase of the power law index $k$ and side-to-thickness ratio $a/h$.

<table>
<thead>
<tr>
<th>$k$</th>
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<th>$a/h=20$</th>
<th>$a/h=40$</th>
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* Results form Ref (Javaheri et al. 2002)

* Results form Ref (Zenkour et al. 2010)
Thermal buckling of functionally graded plates using a n-order four variable refined theory

Fig. 1 Critical buckling temperature difference \( T_{cr} \) due to uniform, linear and non-linear temperature rise across the thickness versus the aspect ratio \( a/b \) \((n=3)\)

Fig. 2 Critical buckling temperature difference \( T_{cr} \) due to uniform, linear and non-linear temperature rise across the thickness versus the side-to-thickness ratio \( a/h \) \((n=3)\)
Fig. 3 Critical buckling temperature difference $T_{cr}$ due to uniform, linear and non-linear temperature rise across the thickness versus the aspect ratio $a/b$ ($n = 3$).

Fig. 4 Critical buckling temperature difference $T_{cr}$ due to uniform, linear and non-linear temperature rise across the thickness versus the side-to-thickness ratio $a/h$ and for different values of the nonlinearity parameter $\gamma$ ($k = 5, n = 3$)

Tables 5 exhibit the critical temperature difference $t_{cr} = 10^{-3}T_{cr}$ for different values of the aspect ratio $a/b$, the temperature exponent $\gamma$ and the power law index $k$ under non-linear temperature loads at $a/h = 10$. The nonlinearity temperature exponent $\gamma$ is taken here as 2, 5 and 10. It can be concluded from the presented results that the present theory gives more accurate results of critical buckling temperature when compared to the higher-order shear deformation theory.
The effect of \( \frac{a}{b} \) on the critical buckling \( t_{cr} \) is similar to that in the case of uniform and linear temperature difference across the thickness. As the power law index \( k \) increases, the critical buckling \( t_{cr} \) decreases to reach lowest values and then increases excluding \( t_{cr} \) of the rectangular plates for \( \gamma = 10 \). It is also noticed from Table 5 that the \( t_{cr} \) increases with the increase of the non-linearity parameter \( \gamma \).

In general, the values of the critical temperature difference calculated by using the shear deformation theories are lower than those calculated by using the classical plate theory, indicating the shear deformation effect.

Figure 1 shows the variation trend of critical temperature difference \( t_{cr} \) with respect to the plate aspect ratio \( \frac{a}{b} \) for different values of material gradient index \( k \) under a uniform, linear and non-linear temperature loads. It is observed that with increasing the plate aspect ratio \( \frac{a}{b} \), the critical buckling temperature difference also increases steadily, whatever the material gradient index \( k \) is. Because the ceramic plate is weaker than the metallic one, thus the critical buckling temperature of the first plate is higher than that of the second. For the FG plate, \( t_{cr} \) decreases as the metallic constituent in the plate increases.

The critical buckling temperature change \( t_{cr} \) versus the side-to-thickness ratio \( \frac{a}{h} \) and the aspect ratio \( \frac{a}{b} \) of FG plates under various thermal loading types is exhibited in Figures 2–4.

It can be seen from these figures that, regardless of the loading type and the power-law index \( k \), the critical buckling temperature difference \( t_{cr} \) decreases as the side-to-thickness ratio \( \frac{a}{h} \) increases and it is reduced with the decrease of the aspect ratio \( \frac{a}{b} \). The critical buckling temperature for the ceramic plate is higher than that for the FG plate. This is because the ceramic plate is stronger than the other. The differences between the loading types decrease with the increase of \( \frac{a}{h} \) because the plate becomes thin. It is also noticed from figure 4 that the \( t_{cr} \) increases with the increase of the non-linearity parameter \( \gamma \).

5. Conclusion

This paper presents a simple n-order four variable refined theory for buckling analysis of functionally graded plates. The present theory is variationally consistent, uses the n-order polynomial term to represent the displacement field, does not require shear correction factor, and eliminates the shear stresses at the top and bottom surfaces. A power law distribution is used to describe the variation of volume fraction of material compositions.

Equilibrium and stability equations are derived based on the present n-order refined theory. The non-linear governing equations are solved for plates subjected to simply supported boundary conditions. The thermal loads are assumed to be uniform, linear and non-linear distribution through-the-thickness.

Based on the above discussion, some conclusions are listed as follows:

- It is shown through the numerical examples that the present theory can provide accurate results for critical temperatures of FG plates subjected to uniformly, linearly and non-linearly distributed temperatures across the thickness.
- The critical buckling temperature difference of FG plates decreases when the side to-
thickness ratio increases $a/h$.

- The critical buckling temperature difference $t_{cr}$ for FG plates is increased by increasing the aspect ratio $a/b$.
- The higher order shear deformation theory underestimates the buckling load compared with the classical plate theory.
- The critical buckling temperature of FG plate under non-linear temperature rise across the thickness increases as the temperature exponent $\gamma$ increases.

References


