Coupled Systems Mechanics, *Vol. 6, No. 4 (2017) 487-499*
DOI: https://doi.org/10.12989/csm.2017.6.4.487

Modeling radon diffusion equation in soil pore matrix by using uncertainty based orthogonal polynomials in Galerkin's method

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(Received August 10, 2017, Revised October 16, 2017, Accepted October 23, 2017)

Abstract. This paper investigates the approximate solution bounds of radon diffusion equation in soil pore matrix coupled with uncertainty. These problems have been modeled by few researchers by considering the parameters as crisp, which may not give the correct essence of the uncertainty. Here, the interval uncertainties are handled by parametric form and solution of the relevant uncertain diffusion equation is found by using Galerkin's Method. The shape functions are taken as the linear combination of orthogonal polynomials which are generated based on the parametric form of the interval uncertainty. Uncertain bounds are computed and results are compared in special cases viz. with the crisp solution.

Keywords: radon; orthogonal; coupled; polynomials; crisp; uncertainty; interval

1. Introduction

Radon is an inert gas with chemical symbol Rn^{222}, and atomic number 86. It is a radioactive, colorless, orderless, tasteless noble gas, occurring naturally as a decay product of uranium. Recent research has shown that radon is the second leading cause of lung cancer. Radon in the soil, groundwater, or building materials is emitted and diffused in the working and living species and then disintegrates into its decay products. So, there is a need to trace the variability of radon levels in different soils. Many experimental researches for soil radon transport have been modelled by diffusion equation through various mediums. There exist variety of physical factors on which radon generation depends viz. radium concentration, porosity and diffusion coefficients which are usually measured experimentally. As such, one may obtain uncertain values or bounds of the parameters rather than exact values. So, the equation describing diffusion of radon in soil pore matrix coupling with uncertain parameters (as intervals) is solved in this work by using the Galerkin's Method with shape functions taken as the linear combination of orthogonal polynomials in uncertain environment.

First, we discuss related important literature based on radon diffusion. The determination of 222 Rn exhalation and effective 226 Ra activity in soil samples are explained by Escobaret *et al.*

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http://www.techno-press.org/?journal=csm&subpage=8 ISSN: 2234-2184 (Print), 2234-2192 (Online)

$$\tilde{p} + \tilde{q} = [\underline{p} + \underline{q},\ \overline{p} + \overline{q}]$$

$$\tilde{p} - \tilde{q} = [\underline{p} - \overline{q}\ \ \overline{p} - \underline{q}]$$

$$\tilde{p} \times \tilde{q} = [\min(\ \underline{p} \times \overline{q}\,,\ \ \underline{p} \times \underline{q},\ \overline{p} \times \underline{q},\ \overline{p} \times \overline{q}),\ \max(\ \underline{p} \times \overline{q}\,,\ \ \underline{p} \times \underline{q},\ \overline{p} \times \underline{q},\ \overline{p} \times \overline{q})\ ]$$

$$\frac{\tilde{p}}{\tilde{q}} = \left[\min\left(\frac{\underline{p}}{\underline{q}}, \frac{\underline{p}}{\overline{q}}, \frac{\overline{p}}{\underline{q}}, \frac{\overline{p}}{\overline{q}} \right), \max\left(\frac{\underline{p}}{\underline{q}}, \frac{\underline{p}}{\overline{q}}, \frac{\overline{p}}{\underline{q}}, \frac{\overline{p}}{\overline{q}} \right) \right],\ \underline{q},\ \overline{q} \neq 0$$

$$\Delta p = \frac{(\overline{p} - \underline{p})}{2} \qquad \text{(Radios of interval p)}$$

$$\tilde{p}_c = \frac{(\overline{p} + \underline{p})}{2} \quad \text{(Center value of interval p)}$$

3. Parametric approach

Parametric approach is used here to represent an interval in crisp form. In this approach, the interval $\tilde{Z} = [\underline{Z},\ \overline{Z}]$ may be written as (Behera and Chakraverty 2015, Tapaswini and Chakraverty 2013)

$\tilde{Z} = \beta(\overline{Z} - \underline{Z}) + \underline{Z}$, where $0 \leq \beta \leq 1$ is a parameter.

It can also be written as

$$\tilde{Z} = 2\beta\,\Delta Z + \underline{Z},\ \ \Delta Z = \frac{(\overline{Z} - \underline{Z})}{2},$$

The lower and upper bounds of the solution can then be obtained by substituting $\beta = 0$ and 1 respectively as follows

$$\tilde{Z} = \underline{Z} \quad \text{when } \beta = 0,$$

$$\tilde{Z} = \overline{Z} \quad \text{when } \beta = 1.$$

4. Galerkin's method for boundary value problems coupled with uncertainty

Let us consider a second order uncertain boundary value problem in [a, b] (Rodriguez 1992)

$$\tilde{a}_1 \tilde{y}'' + \tilde{a}_2 \tilde{y} = q(x) \tag{1}$$

Here $\langle\ ,\ \rangle$ are inner product defined for any two functions as below

$$\langle \phi_i(x), \phi_j(x) \rangle = \int_a^b \phi_i(x)\phi_j(x)dx \tag{3}$$

From Eq. (1) and Eq. (2) we may find the residual '*R*' as

$$R(x; k_{\alpha_i}; k_{0,} k_1 \cdots k_n) = k_{\alpha_1} \sum_{i=0}^{n} k_i \phi_i''(x) + k_{\alpha_2} \sum_{i=0}^{n} k_i \phi_i(x) - q(x) \tag{4}$$

Here the residual R is orthogonalized to the (n+1) functions $\phi_0, \phi_1, \cdots \phi_n$.
This gives

$$\int_a^b R(x; k_{\alpha_i}; k_{0,} k_1 \cdots k_n)\phi_j(x)dx = 0, \quad j = 0,1,2,\cdots n \tag{5}$$

$$\Rightarrow \sum_{i=0}^{n} \left[\int_a^b \left\{ k_{\alpha_1} k_i \phi_i''(x)\phi_j(x) + k_{\alpha_2} k_i \phi_i(x)\phi_j(x) - q(x)\phi_j(x) \right\} dx \right] = 0 \tag{6}$$

Eq. (5) is (n+1) simultaneous equations in (n+1) unknowns, which can be solved by any standard method. Finally, by substituting the evaluated constants $k_0, k_1, k_2 \ldots k_n$ in Eq. (2) we may get the approximate solutions for the uncertain boundary value problem (Eq. (1)) by varying $0 \le \alpha_i \le 1$ for i=1,2,3,4.

5. Radon diffusion mechanism

In pore matrix such as soil, radon is continuously released to the pore volume of the matrix due to the emanation from the grains containing ^{226}Ra. Let us consider that radon diffusion occurs in vertical direction i.e., in x direction after emanating from soil grain to pore space. Let C(x) be the steady state concentration in the soil pore space. Soil properties and radioactivity distributions are assumed to be homogeneous. Then the profile C(x) satisfies the following steady state diffusion equation (Savovic *et al.* 2011, Hafez and Awad 2016)

$$D \frac{\partial^2 C(x)}{\partial x^2} - \lambda C(x) + \lambda C_\infty = 0, \tag{7}$$

where, C(x)= Radon concentration (Bq kg^{-1}) in the soil,
D= The diffusion coefficient of radon in the soil matrix (m^{2}s^{-1}),
λ = The radon decay constant (s^{-1}),
C_∞= The radon concentration when $x \to -\infty$.

The first term and second term of Eq. (7) represents the loss of radon in the pore space of the soil matrix by the process of diffusion and radioactive decay respectively, while the third term represents the production of radon due to emanation from soil grain to pore volume. The boundary

Generating Orthogonal Polynomials (for uncertain radon diffusion equation):

Let us choose $f(x) = (C_0 - K_{\beta_2})e^x + K_{\beta_2}$, which satisfies the boundary conditions involving the parameters in terms of K_{β_2}. We start with two functions, $f_0 = 1,\ f_1 = x$ for two term approximation. Assume that $\phi_0(x),\ \phi_1(x)$ are two orthogonal polynomials (involved uncertainty represented by K_{β_2}) generated by using polynomials $L_0(x),\ L_1(x)$ of the form,

$$L_0(x) = f(x) f_0 = (C_0 - K_{\beta_2})e^x + K_{\beta_2},$$

$$L_1(x) = f(x) f_1 = (C_0 - K_{\beta_2})xe^x + xK_{\beta_2}$$

By using Gram Schmidt orthogonalisation procedure we have

$$\left.\begin{aligned} \phi_0(x) &= L_0(x), \\ \phi_1(x) &= L_1(x) - \delta\phi_0(x) \end{aligned}\right\} \tag{11}$$

where,

$$\Rightarrow \delta = \frac{(C_0 - K_{\beta_2})^2\left(\frac{-1}{4} + \frac{Le^{-2L}}{2} + \frac{e^{-2L}}{4}\right) - \frac{K_{\beta_2}^2 L^2}{2} + 2(C_0 - K_{\beta_2})K_{\beta_2}(-1 + Le^{-L} + e^{-L})}{(C_0 - K_{\beta_2})^2\left(\frac{1}{2} - \frac{e^{-2L}}{2}\right) + K_{\beta_2}^2 L + 2(C_0 - K_{\beta_2}^2)K_{\beta_2}^2(1 - e^{-L})}$$

So, the orthogonal polynomials $\phi_0(x)$ and $\phi_1(x)$ can be represented as

$$\left.\begin{aligned} \phi_0(x) &= (C_0 - K_{\beta_2})e^x + K_{\beta_2}, \\ \phi_1(x) &= (x - \delta)\left((C_0 - K_{\beta_2})e^x + K_{\beta_2}\right). \end{aligned}\right\} \tag{12}$$

Galerkin's Method to solve uncertain radon diffusion equation by using orthogonal polynomials:

We consider two term approximation based on interval uncertainty to approximate the solution of the said diffusion Eq. (9) as

$$\tilde{C}(x) = A_0\phi_0(x) + A_1\phi_1(x), \tag{13}$$

Now, from Eq. (9) and Eq. (13) we have

$$K_{\beta_1}(A_0\phi_0''(x) + A_1\phi_1''(x)) - \lambda(A_0\phi_0(x) + A_1\phi_1(x)) + \lambda K_{\beta_2} = 0 \tag{14}$$

From Eq. (14) the residual '*R*' can be represented as

$$R(x; A_0, A_1, K_{\beta_1}, K_{\beta_2}) = K_{\beta_1}(A_0\phi_0''(x) + A_1\phi_1''(x)) - \lambda(A_0\phi_0(x) + A_1\phi_1(x)) + \lambda K_{\beta_2}$$

Here the residual *R* is orthogonalized to the functions $\phi_0(x), \phi_1(x)$.

This gives

$$\int_{-L}^{0} R(x; A_0, A_1, K_{\beta_1}, K_{\beta_2})\phi_0(x)dx = 0 \tag{15}$$

solutions of the diffusion Eq. (9). The uncertain band of the diffusion equation (Eq. (9)) may be obtained by varying the values of $\beta_1, \beta_2 \in [0,1]$

7. Results and discussions

Here, the results are presented based on radon diffusion equation (Eq. (9)) solved by Galerkin's Method using uncertainty based orthogonal polynomials. A soil pore matrix is considered with depth (L=10m), in which the radon diffusion occurs in vertical direction x. It is assumed that the initial radon concentration in soil pore matrix at $x = 0$ as $C_0 = 10(Bq/m^3)$ and radon concentration at $x = -L$ is supposed to be exposed to high radon concentration $C_\infty = 1000(Bq/m^3)$. The value $D = 2.1\times10^{-6}\,(m^2/s)$ was used for the radon diffusion coefficient in soil and the decay constant (λ) of radon taken as $2.1\times10^{-6}\,s^{-1}$.

Table 1 lists the numerical values of the involved crisp and interval parameters when the radon diffusion equation is coupled with interval uncertainty.

Table 1 Numerical values for involved parameters of uncertain based diffusion equation (Savovic *et al.* 2011)

Parameter	Crisp Value	Interval Value
C_∞	$1000(Bq/m^3)$	$[980, 1020](Bq/m^3)$
D	$2.1\times10^{-6}\,(m^2/s)$	$[1.5\times10^{-6}, 2.7\times10^{-6}]$
C_0	$10(Bq/m^3)$	$10(Bq/m^3)$
λ	$2.1\times10^{-6}\,s^{-1}$	$2.1\times10^{-6}\,s^{-1}$

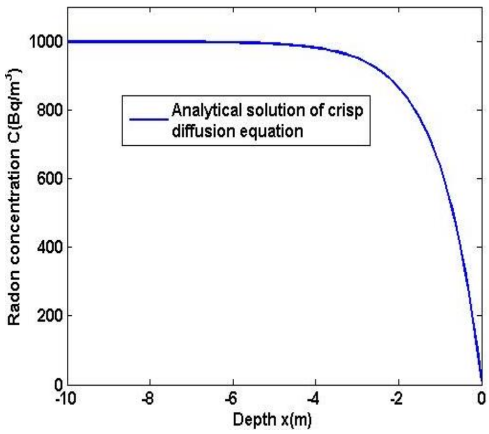

Fig. 1 Analytical solution of crisp diffusion equation

Fig. 1 presents the analytical solution of the crisp radon diffusion Eq. (7). Fig. 2 depicts the comparison of analytical solution of the crisp diffusion equation (Eq. (7)) with the center solution obtained by solving the uncertain diffusion equation (Eq. (9)) by using Galerkin's Method (when

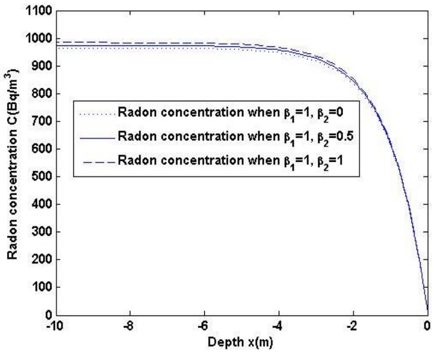

Fig. 5 Lower, center, upper radon concentration when $\beta_1 = 1,\ \beta_2 = 0:0.5:1$

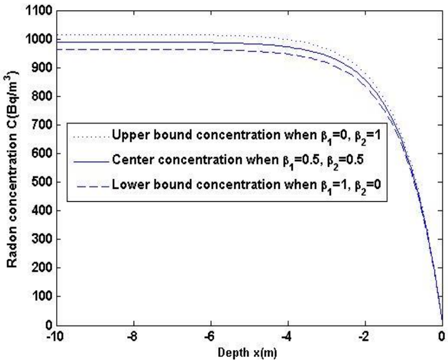

Fig. 6 Lower, upper bounds and center radon concentration from all the possible combinations $\beta_1 = 0:0.5:1,\ \ \beta_2 = 0:0.5:1$

Table 2 presents the different radon concentrations for a typical fixed depths ($x=-1, x=-5, x=-10$) by varying $0 \le \beta_1, \beta_2 \le 1$.

Table 2 Radon concentration for different combinations of $\beta_1, \beta_2 \in [0,1]$ at fixed depths $(x=-1, x=-5, x=-10)$

Parametric values		Concentration		
β_1	β_2	$x=-1$	$x=-5$	$x=-10$
0	0	633.2081	989.1446	995.7145
	0.4	638.3467	997.2189	1003.8
	0.8	643.4854	1005.3	1012
0.4	0	625.1409	976.5602	983.0687
	0.4	630.2139	984.5316	991.0938
	0.8	635.2870	992.5030	999.1188
0.8	0	617.2813	964.2931	970.7332
	0.4	622.2905	972.1642	978.6574
	0.8	627.2997	980.0353	986.5817

From the above presented Table, one may observe the increase of radon concentration with respect to the depth of the soil and the effect of diffusion coefficient (for the lower values of diffusion coefficient concentration giving high).

8. Conclusions

In this paper, we presented a new approach to solve radon diffusion equation when coupled with uncertainty (Eq. (9)). Here the approximate solution of the same is assumed first as a linear combination of orthogonal polynomials with interval uncertainty. The involved parameters with interval uncertainty are represented by using the parametric concept. Then the uncertain radon diffusion equation has been solved by using Galerkin's Method. Finally, we depicted different uncertainty bands of radon diffusion equation (Eq. (9)) by varying $\beta_1, \beta_2 \in [0,1]$. For the validation, the results are compared with the known analytical solution and are found to be in good agreement.

Acknowledgments

The authors are thankful to Board of Research in Nuclear Sciences (BRNS), Mumbai, India (Grant Number: 36(4)/40/46/2014-BRNS) for the support and funding to carry out the Present research work.

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