Settlement analysis of viscoelastic foundation under vertical line load using a fractional Kelvin-Voigt model

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Abstract. Soil foundations exhibit significant creeping deformation, which may result in excessive settlement and failure of superstructures. Based on the theory of viscoelasticity and fractional calculus, a fractional Kelvin-Voigt model is proposed to account for the time-dependent behavior of soil foundation under vertical line load. Analytical solution of settlements in the foundation was derived using Laplace transforms. The influence of the model parameters on the time-dependent settlement is studied through a parametric study. Results indicate that the settlement-time relationship can be accurately captured by varying values of the fractional order of differential operator and the coefficient of viscosity. In comparison with the classical Kelvin-Voigt model, the fractional model can provide a more accurate prediction of long-term settlements of soil foundation. The determination of influential distance also affects the calculation of settlements.

Keywords: soil foundation; fractional viscoelastic model; the Flamant-Boussinesq solution; settlement; Laplace transform.

1. Introduction

Research results show that the soil foundation exhibits significant creeping behaviour and the ground deformation under loading from superstructures is highly time-dependent (Bjerrum 1967). For the estimation of the increase in stresses at various points and associated displacements caused in soil foundations due to external loading, there have been a number of viscoelastic, elastic-visco-plastic (EVP) or elastoplastic-viscoplastic models (e.g., Christie 1964, Kaliakin and Dafalias 1990, Yin and Graham 1994, Justo and Durand 2000). Among them, viscoelastic foundation models can provide satisfactory simulation of the time-dependent behaviour when the stress level in soil is low and yielding of soil is thus not reached. The ideal assumption of the theory of viscoelasticity, namely that the foundation is homogeneous, viscoelastic, and isotropic, is not quite true in most cases. However, it provides a close estimation of long-term settlement and failure possibility of superstructures and, using proper safety factors, a safe design can be developed.

Because integer-order differential operators are used in the viscoelastic constitutive equations, the

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kernel functions (creep modulus and relaxation modulus) of viscoelastic models are normally a combination of exponential functions. Thus these models have difficulties in predicting settlement of soil foundation accurately and their applicable range is significantly limited. Fractional constitutive models of viscoelastic materials were first introduced by Gemant (1936). In the constitutive equations, integer-order differential operators were replaced by fractional-order differential operators. These models have shown powerful capability in capturing static and dynamic stress-strain-time relationships of viscoelastic materials and have been applied in various fields in the past few decades (Bagley and Torvik 1983 and 1986, Welch *et al.* 1999, Zhang and Shimizu 1998, Li *et al.* 2001, Rossikhin and Shitikova 2010). However, due to the complexity involved in obtaining the analytical solution, these fractional models have not been applied in the field of geotechnical engineering until recent years (Atanackovic and Stankovic 2004, Dikmen 2005, Liu *et al.* 2006, Paola *et al.* 2009, Zhu *et al.* 2011).

In this paper, a fractional Kelvin-Voigt viscoelastic constitutive model was proposed to study time-dependent settlement of soil foundations. The analytical solution of a half-space under a vertical line load is derived using Laplace transforms. The influences of the model parameters and the predetermined influential distance on the calculated settlements at the foundation surface are further explored using a parametric study.

2. Fractional Kelvin-Voigt model

The generalized viscoelastic constitutive equation using fractional derivatives can be described by (Padovan 1987)

$$\sigma(t) + \sum_{m=1}^{M} b_m \frac{d^{\beta_m}}{dt^{\beta_m}}\sigma(t) = E_0\varepsilon(t) + \sum_{n=1}^{N} E_n \frac{d^{\alpha_n}}{dt^{\alpha_n}}\varepsilon(t)\ (0<\beta_m<1, 0<\alpha_n<1) \tag{1}$$

where m and n are positive integers; d^{β_m}/dt^{β_m} and $d^{\alpha_n}/dt^{\alpha_n}$ are Riemann-Liouville fractional differential operators defined by (Miller and Ross 1993)

$$\frac{d^{\alpha}\mathrm{f}(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\mathrm{f}(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau\ \ (0<\alpha<1) \tag{2}$$

where $\Gamma(u)$ is the Gamma function, $\Gamma(u) = \int_0^{\infty} t^{u-1}e^{-u}dt$.

If $m = n = 1$ and $b_1 = 0$, Eq. (1) can be expressed by

$$\sigma(t) = E_0\varepsilon(t) + E_1\frac{d^{\alpha}\varepsilon(t)}{dt^{\alpha}} \tag{3}$$

It is obvious that if $\alpha = 1$, Eq. (3) represents the constitutive equation of the classical Kelvin-Voigt viscoelastic model (KVM). Thus this new model can be called a generalized fractional Kelvin-Voigt model (FKVM), in which the integer-order differential operator is now replaced by a fractional-order one.

In the theory of viscoelasticity, the stress-strain relationships are

$$P'S_{ij}(t) = Q'e_{ij}(t) \tag{4}$$

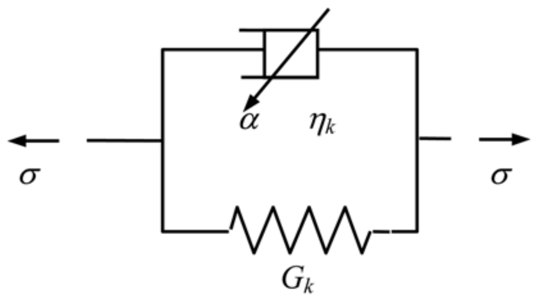


Fig. 1 Three-parameter fractional Kelvin-Voigt model

$$P''\sigma_{ij}(t) = Q''\varepsilon_{ij}(t) \tag{5}$$

where $S_{ij}(t)$ and $\sigma_{ij}(t)$ are deviatoric and hydrostatic stress tensors; $e_{ij}(t)$ and $\varepsilon_{ij}(t)$ are deviatoric and hydrostatic strain tensors; P', Q', P'' and Q'' are linear differential operators consisting of viscoelastic parameters.

The volumetric and shear strains of foundation soils under low stress levels are assumed to be time-dependent. The FKVM shown in Fig. 1 is thus utilized for the description of the viscoelastic behaviour of the bulk and shear moduli,, which describe the volume and shape changes of soil mass under external loading. Thus

$$P' = 1 \tag{6}$$

$$Q' = G + \eta\frac{d^\alpha}{dt^\alpha} \tag{7}$$

$$P'' = 1 \tag{8}$$

$$Q'' = K + \eta\frac{d^\alpha}{dt^\alpha} \tag{9}$$

where G, K and η are the shear modulus, the bulk modulus, and the coefficient of viscosity, respectively. Determination of the model parameters can be obtained from laboratory and field testing.

3. Analytical solutions of settlements using fractional Kelvin-Voigt model

3.1 Flamant-Boussinesq elastic solution

Fig. 2 shows the case where a line force of p_0 per unit length is applied on the surface of a half-space. The elastic solution for this plane-strain problem was developed by Flamant (1892) by modifying the three-dimensional solution of Boussinesq (Timoshenko and Goodier 1970). The Flamant-Boussinesq solution using an elastic model (EM) is widely used in foundation engineering for the calculation of stresses and estimation of displacements.

In the polar coordinate system, assume that at a point P(r_0, $\pi/2$) on the ground surface, $(u_\theta)_{\theta=\frac{\pi}{2}, r=r_0} = 0$, i.e., the influential distance of ground surface settlement is assumed to be r_0. The displacements at any point in the foundation can be obtained as

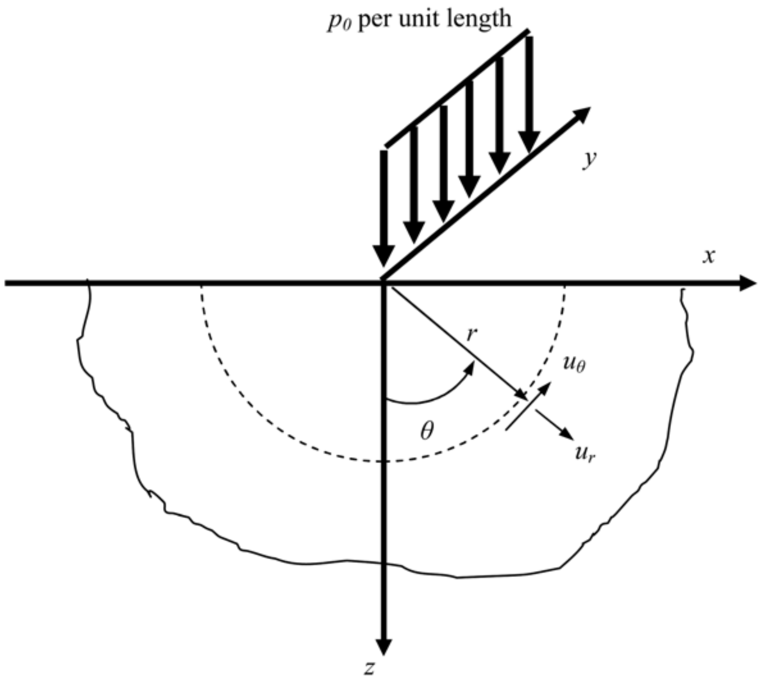


Fig. 2 Schematic illustration of a foundation subjected to vertical line load at the surface

$$u_r = \frac{2p_0(1-\mu^2)}{\pi E}\cos\theta \ln\frac{r_0}{r} + \frac{(1-2\mu)(1+\mu)p_0}{\pi E}\theta\sin\theta - \frac{(1+\mu)p_0}{\pi E}\cos\theta \tag{10}$$

$$u_\theta = -\frac{2p_0(1-\mu^2)}{\pi E}\sin\theta \ln\frac{r_0}{r} + \frac{(1-2\mu)(1+\mu)p_0}{\pi E}\theta\cos\theta \tag{11}$$

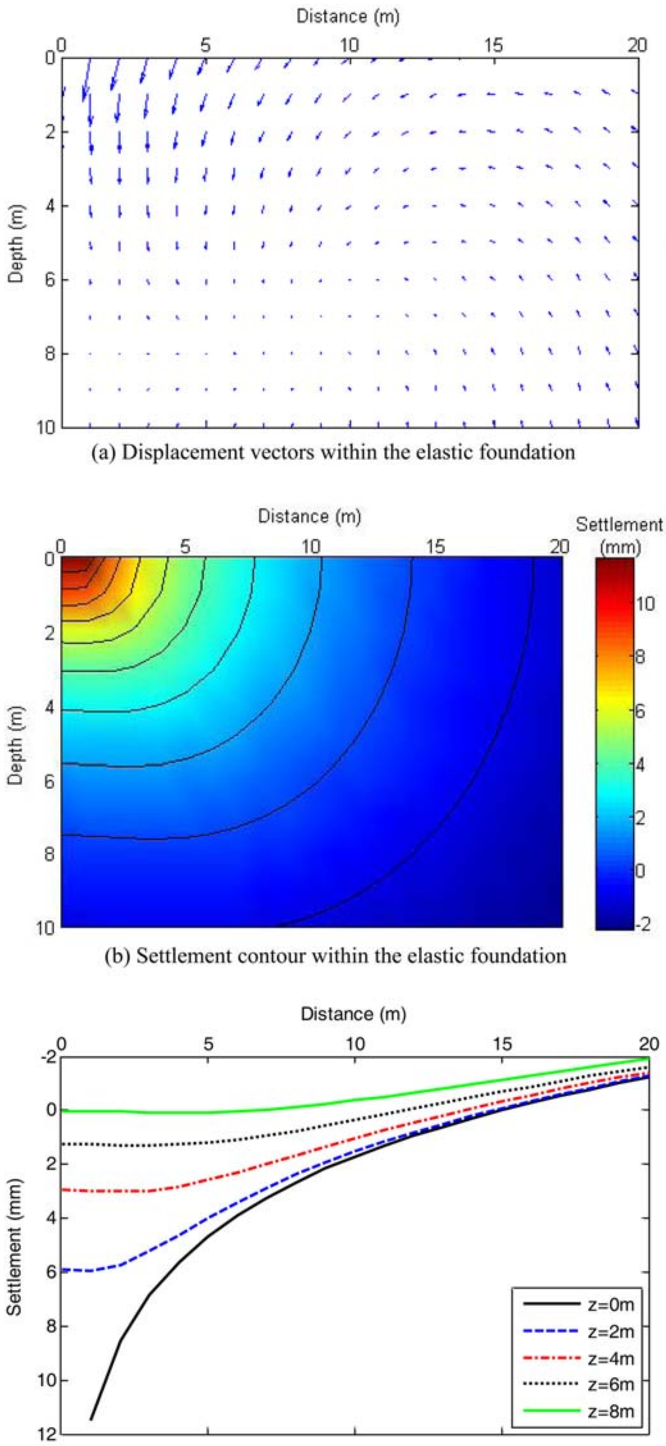
where E and μ are the modulus of elasticity and Poisson's ratio, respectively; u_r and u_θ denote the displacements in the radial and circumferential directions, respectively.

Taking $E = 144$ MPa, $\mu = 0.2$, $p_0 = 1$ MPa and $r_0 = 15$ m, the distributions of displacements within the soil foundation using the elastic model (EM) are depicted in Fig. 3. It is seen that the surface settlement at the loading point approaches infinity and the settlements within the foundation reduce sharply with increasing depth. For locations far away from the loading point, heave of the foundation occurs.

In the case of $\theta = \pi/2$, Eqs. (10) and (11) for radial and circumferential displacement components at the ground surface can be given as the following simple form

$$(u_r)_{\theta=\frac{\pi}{2}} = \frac{(1-2\mu)(1+\mu)p_0}{2E} = -\frac{3p_0}{4(3K+G)} \tag{12}$$

$$(u_\theta)_{\theta=\frac{\pi}{2}} = \frac{2p_0(1-\mu^2)}{\pi E}\ln\frac{r_0}{r} = -\frac{p_0\ln\frac{r_0}{r}}{2\pi}\left[\frac{3}{(3K+G)}+\frac{1}{G}\right] \tag{13}$$



(a) Displacement vectors within the elastic foundation

(b) Settlement contour within the elastic foundation

(c) Settlement distributions at different depths

Fig. 3 Displacement distributions calculated from the Flamant-Boussinesq elastic solution

Eqs. (12) and (13) indicate that there is a logarithmic relationship between the distance to the loading point and the vertical displacement, and that the horizontal displacement at the ground surface is a constant that is not affected by *r*.

3.2 Viscoelastic solution using classical Kelvin-Voigt model

Considering a line load applied on the foundation shown in Fig. 2, the loading condition is specified as

$$p = p_0 H(t) \tag{14}$$

where $H(t)$ is the unit-step function defined as $H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$.

To obtain the viscoelastic solution, the volumetric and shear strains of the foundation soil satisfies the KVM. Thus Eqs. (7) and (9) turns into

$$Q' = G + \eta \frac{d}{dt} \tag{15}$$

$$Q'' = K + \eta \frac{d}{dt} \tag{16}$$

Taking the Laplace transforms of Eqs. (6), (8), (15) and (16), and combining them with Eqs. (12) and (13), the displacements at the ground surface can be expressed by

$$(\bar{u}_r)_{\theta=\frac{\pi}{2}} = -\frac{3p_0}{4} \frac{1}{(3K+G+4\eta s)s} \tag{17}$$

$$(\bar{u}_\theta)_{\theta=\frac{\pi}{2}} = -\frac{p_0 \ln\frac{r_0}{r}}{2\pi}\left[\frac{3}{(3K+G+4\eta s)s} + \frac{1}{(G+\eta s)s}\right] \tag{18}$$

Taking the inverse Laplace transforms of Eqs. (17) and (18), we get

$$(u_r)_{\theta=\frac{\pi}{2}} = -\frac{3p_0}{4(3K+G)}\left(1 - e^{-\frac{t}{\tau_1}}\right) \tag{19}$$

$$(u_\theta)_{\theta=\frac{\pi}{2}} = -\frac{p_0 \ln\frac{r_0}{r}}{2\pi}\left[\frac{3}{(3K+G)}\left(1 - e^{-\frac{t}{\tau_1}}\right) + \frac{1}{G}\left(1 - e^{-\frac{t}{\tau_2}}\right)\right] \tag{20}$$

where $\tau_1 = 4\eta/(3K+G)$, $\tau_2 = \eta/G$. Eqs. (19) and (20) show that the time-dependent displacements are exponential creep functions. It is obvious that if $t \to \infty$, the ultimate displacements equal the Flamant-Boussinesq solution using elastic model (EM).

3.3 Viscoelastic solution using fractional Kelvin-Voigt model

Taking the Laplace transforms of Eqs. (6) to (9) and combined with Eqs. (12) and (13), the displacements at the ground surface can be expressed by

$$(\bar{u}_r)_{\theta=\frac{\pi}{2}} = -\frac{3p_0}{4}\frac{1}{(3K+G+4\eta s^{\alpha})s} \tag{21}$$

$$(\bar{u}_\theta)_{\theta=\frac{\pi}{2}} = -\frac{p_0 \ln\frac{r_0}{r}}{2\pi}\left[\frac{3}{(3K+G+4\eta s^{\alpha})s} + \frac{1}{(G+\eta s^{\alpha})s}\right] \tag{22}$$

In order to take the inverse Laplace transform, note the following characteristics of the fractional derivative in the Laplace transform for zero initial condition

$$L^{-1}\left[\frac{1}{(G+\eta s^{\alpha})s}\right] = \frac{1}{G}\left\{1 - E_{\alpha}\left[-\left(\frac{t}{\tau}\right)^{\alpha}\right]\right\} \tag{23}$$

where E_α is the Mittag-Leffler function defined as $E_\alpha(t) = \sum_0^\infty \frac{t^n}{\Gamma(1+\alpha n)}$; $\tau = \frac{\eta}{G}$.

In view of Eq. (23), taking the inverse Laplace transforms of Eqs. (21) and (22), we get

$$(u_r)_{\theta=\frac{\pi}{2}} = -\frac{3p_0}{4(3K+G)}\left\{1 - E_{\alpha}\left[-\left(\frac{t}{\tau_1}\right)^{\alpha}\right]\right\} \tag{24}$$

$$(u_\theta)_{\theta=\frac{\pi}{2}} = -\frac{p_0 \ln\frac{r_0}{r}}{2\pi}\left\{\frac{3 - 3E_{\alpha}\left[-\left(\frac{t}{\tau_1}\right)^{\alpha}\right]}{(3K+G)} + \frac{1 - E_{\alpha}\left[-\left(\frac{t}{\tau_2}\right)^{\alpha}\right]}{G}\right\} \tag{25}$$

where $\tau_1 = \frac{4\eta}{3K+G}$, $\tau_2 = \frac{\eta}{G}$.

It is obvious that when $0 < \alpha \le 1$, $\lim_{t\to\infty}(u_r)_{\theta=\frac{\pi}{2}} = -\frac{3p_0}{4(3K+G)}$, $\lim_{t\to\infty}(u_\theta)_{\theta=\frac{\pi}{2}} = -\frac{p_0 \ln\frac{r_0}{r}}{2\pi}\left[\frac{3}{(3K+G)} + \frac{1}{G}\right]$.

Thus the ultimate displacements equal the elastic solution.

If $\alpha \to 0$, $E_\alpha(t) \to \sum_0^\infty \frac{t^n}{\Gamma(1)} = \sum_0^\infty t^n = \frac{1}{1-t}(-1 < t < 1)$. Thus Eqs. (24) and (25) turn into

$$(u_r)_{\theta=\frac{\pi}{2}} = -\frac{3p_0}{8(3K+G)} \tag{26}$$

$$(u_\theta)_{\theta=\frac{\pi}{2}} = -\frac{p_0 \ln\frac{r_0}{r}}{4\pi}\left[\frac{3}{3K+G} + \frac{1}{G}\right] \tag{27}$$

Eqs. (26) and (27) indicate that the displacements at the ground surface are not time-dependent any more and the displacement values are half of those from the elastic solution. Therefore, the FKVM collapses to be a ideal elastic model as $\alpha \to 0$.

If $\alpha = 1$, $E_\alpha(t) = \sum_0^\infty \frac{t^n}{\Gamma(1+\alpha n)} = \sum_0^\infty \frac{t^n}{n!} = e^t$. Obviously, Eqs. (23) and (24) currently become the viscoelastic solution, which means that the FKVM turns into a classical KVM.

4. Numerical example and analysis

4.1 Comparison of elastic and viscoelastic models

How to determine the fractional model parameters is a major concern in practical applications. To carry out a preliminary investigation of these parameters' influence on the calculation results, a parametric study is presented as following. Here, we analyze the quasistatic problem of a viscoelastic foundation subjected to line loading $p_0 = 1$ MPa using the FKVM. The basic parameters used in this analysis are listed in Table 1.

Fig. 4 shows the surface settlement results using the fractional model at $t = 10$ d and 50 d, together with the results from the classical KVM solution and the Flamant-Boussinesq elastic solution. The present results show that the discrepancy between the results from FKVM and KVM increases as t increases. The calculated settlements using KVM develops rapidly with the elapsed time while those using FKVM will take much more time.

4.2 Parametric study of the FKVM

To study the influence of the coefficient of viscosity in the FKVM, η is set to 500, 1000 and 1500 MPa·d. Fig. 5 depicts the settlement-time curves at $r = 1$ m over time using the FKVM, the KVM and the EM. It is shown that the coefficient of viscosity mainly affects the overall creeping rate of the foundation soil. For the same time point, the settlements increase with the decrease of η. Note that the ultimate settlement in the viscoelastic solution is equal to that in the elastic solution, η may determine the magnitude of creeping by controlling the time duration to achieve the ultimate settlement. If η decreases, the creeping effect will function more slowly and the ultimate settlement will be obtained for a much longer time. Results from the classical Kelvin-Voigt model (α = 1), which is shown in Fig. 6 by a dashed line, shows larger settlements and less time to get to the ultimate settlement.

Table 1 Parameters of the FKVM

Item	Value
Young's modulus E (MPa)	144
Poisson's ratio μ	0.2
Shear modulus G (MPa)	60
Bulk modulus K (MPa)	80
Coefficient of viscosity η (MPa·d)	1000
Fractional differential order α	0.5
r_0 (m)	15

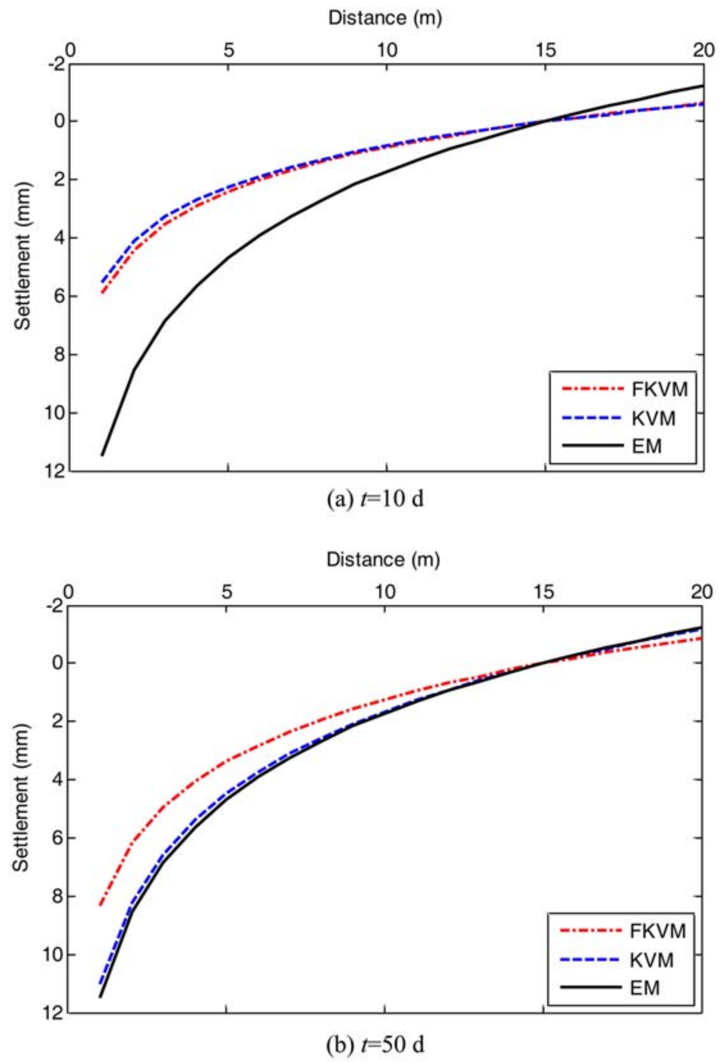


Fig. 4 Comparsion of settlements calculated using elastic, Kelvin-Voigt and fractional Kelvin-Voigt models

To investigate the influence of the fractional order of differential operator in the FKVM, α is set to 0.1, 0.3, 0.5, 0.7 and 0.9 while η is fixed to 1000 MPa·d. Fig. 6 illustrates the settlement at $r = 1$ m over time. It is shown that the fractional order α has a significant influence on the long-term performance of the foundation surface settlements. At $t_{1/2} = 10$ d, the settlements are half of the ultimate settlement (5.75 mm) for any value of α. $t_{1/2}$ is found to have an approximately linear relationship with η and $1/E$. After that time point, with the increase of α, the foundation settlement and corresponding settlement rate increase. Therefore, α may determine the functioning law of creeping effect by separating the time-dependent deformation of soil foundation into two stages. The first stage may associate with the consolidation process of the foundation soil and the second stage is mainly due to secondary compression. Therefore, a small α value is fit for sand foundation, which has a small permeability coefficient. While for soft clay foundation, the α value should be

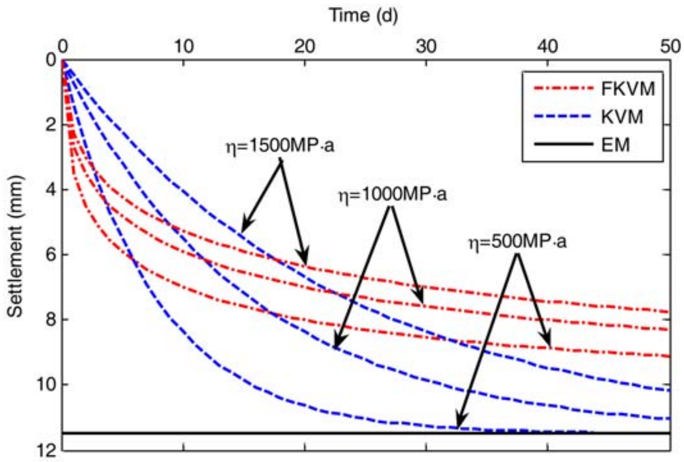


Fig. 5 Settlement-time relationship at r = 1 m under different η values

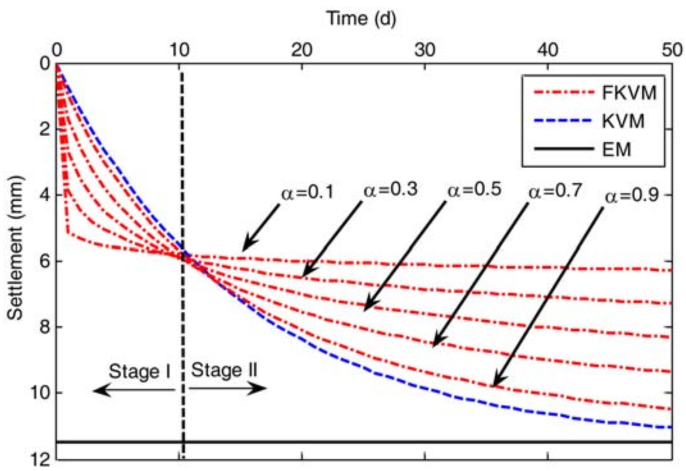


Fig. 6 Settlement-time relationship at r = 1 m under different α values

increased. In the KVM where α = 1 (the dashed line in Fig. 6), a minimum settlement rate at stage I and a maximum settlement rate at stage II is obtained.

The key feature of the FKVM illustrated in Figs. 5 and 6 is the capability to represent generalised viscoelastic behaviour over many decades of time with only three parameters. It is also indicated from the above parametric study that the classical viscoelastic model may overestimate the development of long-term settlement (Stage II) and fail to describe the attenuation creep of soil mass accurately. Using the fractional viscoelastic models, we can obtain the best fitting effect of the actual soil deformation characteristics by selecting appropriate values of the fractional differential order and the coefficient of viscosity. In geotechnical design, this model can provide more accurate settlement prediction and thus save construction costs of foundations or embankments.

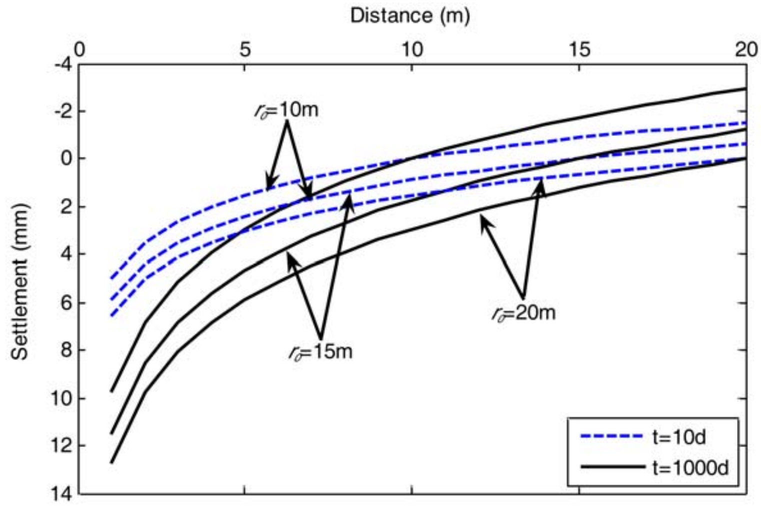


Fig. 7 Settlement distributions at the foundation surface under different predetermined r_0 values

4.3 Determination of the influential distance

The determination of influential distance due to loading is another major engineering concern. Fig. 7 shows the settlement results when the value of r_0 is set to 10, 15 and 20 m while α is fixed to be 0.5. It is shown that with the increase of r_0, the time-dependent and ultimate settlement (t = 1000 d) will increase but the settlement profile is not affected significantly. If the influential distance of settlement is overestimated (like r_0 = 20 m), a larger settlements will be predicted.

5. Conclusions

Based on the theory of viscoelasticity and fractional calculus, a fractional Kelvin-Voigt viscoelastic model is developed to account for the time-dependent behaviour of soil foundations under concentrated line load. Analytical solution of displacements in the foundation was derived using Laplace transforms. The fractional theory shows its potential in modeling the long-term foundation deformation. The presented analytical procedure can be extended to solve other geotechnical problems. From a parametric study, the fractional order of derivative differential operator and the coefficient of viscosity exert a significant influence on the time-dependent settlements of foundation. Compared with the KVM, the FKVM can accurately capture the actual deformation of soil foundations under loading.

However, the empirical relationship between the model parameters and soil properties is an important issue to be solved, which requires the support of abundant measurement data from laboratory and field experiments. Careful attention should be paid to the predetermined influential distance, as well.

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